Function Spaces
Mid Semester Exam
Marks - 30, Duration - 2 Hours

1. [2 marks] Let (M, d) be a metric space and let $A \subseteq M$. For $x \in M$, define $d(x, A) = \inf\{d(x, y) : y \in A\}$. Pick out the true statements:

   (a) $x \mapsto d(x, A)$ is a uniformly continuous function.

   (b) If $\operatorname{bdry} A = \{x \in M : d(x, A) = 0\} \cap \{x \in M : d(x, A^c) = 0\}$, then $\operatorname{bdry} A$ is closed for any $A \subseteq M$.

   (c) For two disjoint closed sets A and B, $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\} > 0$.

   (d) For two disjoint compact sets A and B, $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\} > 0$.

2. [2 marks] Pick out the compact sets.

   (a) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\} \subset \mathbb{R}^2$.

   (b) $\{\operatorname{Tr}(A) : A \in M(n; \mathbb{R}), A \text{ is orthogonal}\} \subset \mathbb{R}$, where $\operatorname{Tr}(A)$ denotes the trace of the matrix A.

   (c) The set of all matrices in $M(n; \mathbb{R})$ all of whose eigenvalues satisfy the condition $|\lambda| \leq 2024$.

   (d) $\{\boldsymbol{e}_i = (0, \ldots, 0, \underbrace{1}_{i^{\text{th}} \text{ place}}, 0, \ldots) : i \geq 1\} \subset l_\infty$.

3. [2 marks] Which of the following statements are true?

   (a) The sequence
   $$\{f_n\}_{n\geq 1}, \quad \text{where } f_n(x) = \frac{nx}{1 + nx},$$
   converges uniformly on $x \in (1, \infty)$.

   (b) The series
   $$\sum_{n=1}^{\infty} \frac{x}{n^{2024}(1 + nx^2)}$$
   converges uniformly on $x \in [2023, 2025]$.

   (c) The sequence
   $$\{f_n\}_{n\geq 1}, \quad \text{where } f_n(x) = \frac{x^n}{1 + x^n},$$
   converges uniformly on $x \in [0, 2]$.

   (d) The sum of the series
   $$\sum_{n=1}^{\infty} \frac{\sin nx^2}{1 + n^3}$$
   defines a continuously differentiable function on $\mathbb{R}$.

4. [3 marks] Let (M, d) be a compact metric space and let $\{f_n\}_{n\geq 1}$ be a sequence in $C(M)$ that converges uniformly to $f \in C(M)$. Prove that $\{f_n\}_{n\geq 1}$ is uniformly bounded and equicontinuous.

5. [3 marks] Show that the map T on $(C[0,1], \|.\|_\infty)$ defined by

$$Tf(x) := \int_0^x (x-t)f(t)\mathrm{dt}, \quad 0 \le x \le 1, \quad f \in C[0,1],$$

is a contraction. What is its fixed point?

6. [4 marks] A real-valued function f on a metric space (M, d) is said to be upper semi-continuous if for each $\alpha \in \mathbb{R}$, the set $\{x \in M : f(x) < \alpha\}$ is open in M. Prove that if M is compact, then every upper semi-continuous function on M is bounded above and attains a maximum value.

7. [4 marks] Let $\{f_n\}_{n\ge1}$ be a sequence in $C[0,1]$. If the sequence $\{f_n\}_{n\ge1}$ converges to f uniformly on $[0,1]$, show that

$$\lim_{\substack{n\to\infty \\ n\ge 2025}} \int_{\frac{2024}{n}}^{1-\frac{2024}{n}} f_n(t)dt = \int_0^1 f(t)dt.$$

8. [4 marks] Let $\{f_n\}_{n\ge1}$ be a sequence of functions defined by

$$f_1(x) = \sqrt{x}, \quad f_{n+1}(x) = \sqrt{xf_n(x)}, \text{ for all } n \ge 1.$$

Prove that the sequence $\{f_n\}_{n\ge1}$ converges uniformly on $[0,1]$, and find the limit function.

9. [6 marks] Let (M, d) be a metric space and let $f : M \to M$. If f^n is a weak contraction, for some $n \ge 1$, and $\overline{f^n(M)}$ is compact, then prove that f has a unique fixed point $\bar{x} \in M$.